

Castigliano's Theorem Applied to Beams

- The internal strain energy for a beam is caused by both bending and shear.
- Castigliano's theorem for beams states that

$$\Delta = \int_0^L M \left(\frac{\partial M}{\partial P} \right) \frac{dx}{EI}$$

Δ = displacement of the point

P = external force

M = internal moment in the beam

E = modulus elasticity of the material

I = moment of inertia

- If the slope of the tangent at a point on the elastic curve is to be determined,

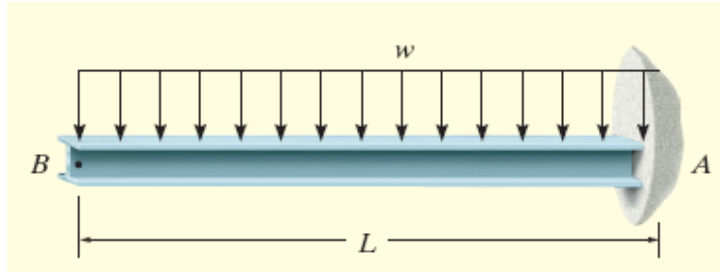
$$\theta = \int_0^L M \left(\frac{\partial M}{\partial M'} \right) \frac{dx}{EI}$$

M' = external couple moment



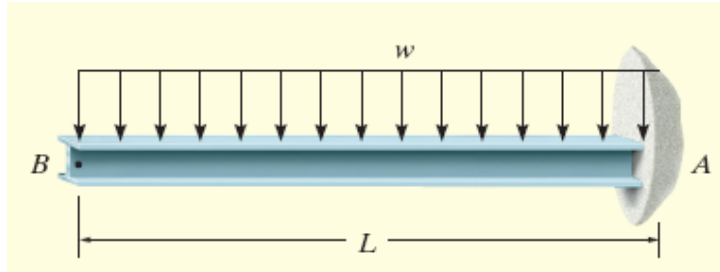
Example 3

Determine the displacement of point B on the beam. EI is constant.



Solution:

Determine the displacement of point B on the beam. EI is constant.



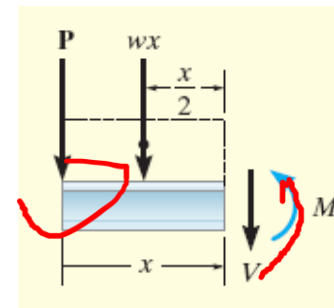
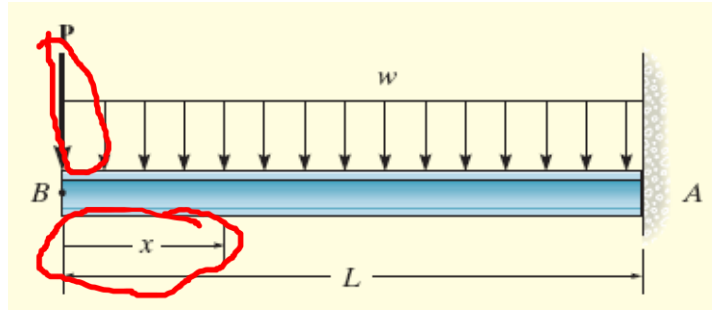
A vertical force P is placed on the beam at B .

Using the method of sections,

$$+\sum M_{NA} = 0; \quad M + wx\left(\frac{x}{2}\right) + P(x) = 0$$

$$M = -\frac{wx^2}{2} - Px$$

$$\frac{\partial M}{\partial P} = -x$$



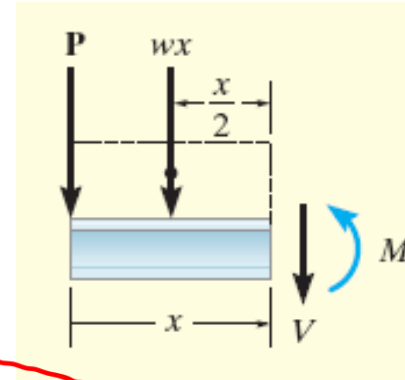
Solution:

Setting $P = 0$ gives

$$M = \frac{-wx^2}{2} \text{ and } \frac{\partial M}{\partial P} = -x$$

Using Castigliano's Second Theorem,

$$\Delta_B = \int_0^L M \left(\frac{\partial M}{\partial P} \right) \frac{dx}{EI} = \int_0^L \frac{(-wx^2/2)(-x)dx}{EI} = \frac{wL^4}{8EI} \quad (\text{Ans})$$



Example 4

Determine the vertical displacement of point C of the steel beam.

Take $E_{st} = 200 \text{ GPa}$, $I = 125(10^{-6}) \text{ m}^4$.

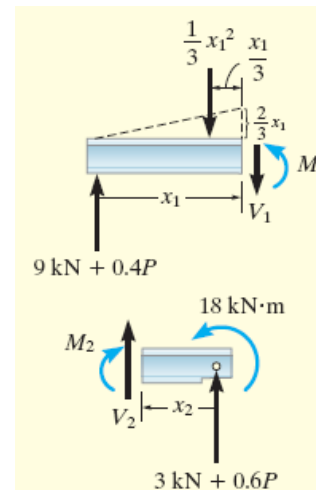
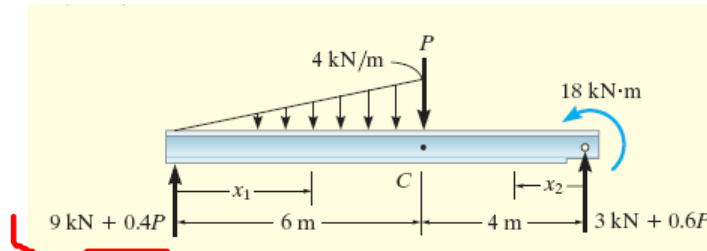
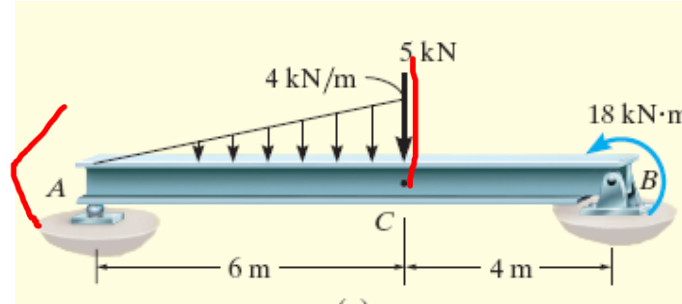
A vertical force P is placed on the beam at C.

Using the method of sections, for x_1 and x_2 ,

$$+\sum M_{NA} = 0; \quad M_1 + \frac{1}{3}x_1^2\left(\frac{x_1}{2}\right) - (9 + 0.4P)x_1 = 0$$

$$M_1 = (9 + 0.4P) - \frac{1}{9}x_1^3$$

$$\frac{\partial M_1}{\partial P} = 0.4x_1$$



Solution:

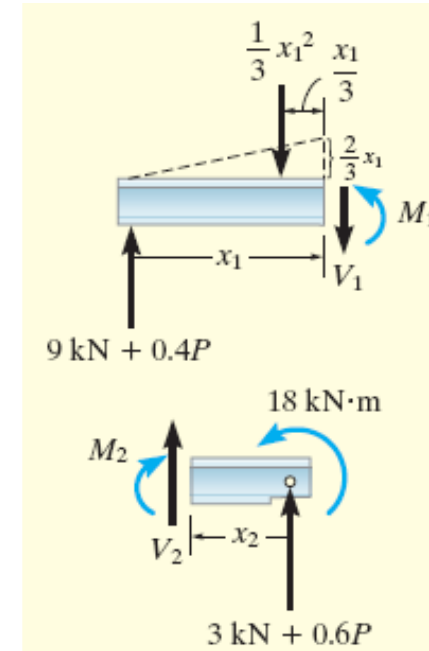
$$+\sum M_{NA} = 0; \quad -M_2 + 18 + (3 + 0.6P)x_2 = 0$$

$$M_2 = 18 + (3 + 0.6P)x_2$$

$$\frac{\partial M_2}{\partial P} = 0.6x_2$$

Using Castigliano's Second Theorem by setting $P = 5\text{ kN}$,

$$\begin{aligned} \Delta_{C_v} &= \int_0^L M \left(\frac{\partial M}{\partial P} \right) \frac{dx}{EI} \\ &= \int_0^6 \frac{(11x_1 - \frac{1}{9}x_1^3)(0.4x_1)dx_1}{EI} + \int_0^4 \frac{(18 - 6x_2)(0.6x_2)dx_2}{EI} \\ &= \frac{410.9}{[200(10^6)]125(10^{-6})} = 0.0164 \text{ m} = 16.4 \text{ mm} \quad (\text{Ans}) \end{aligned}$$





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